Pergamon

Int. J. Solids Structures Vol. 35, No. 23, pp. 3043–3052, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00354–5

METHOD OF TRANSLATIONS FOR A MODE I ELLIPTIC CRACK IN AN INFINITE BODY. PART II: EXPANSION OF THE FUNDAMENTAL SOLUTION INTO A SERIES

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(*Received* 15 *January* 1997; *in revised form* 3 *November* 1997)

Abstract—The present paper considers the consequences of the application of the theory of crack translation in a nonuniform stress field to the analysis of an elliptic crack in an infinite body, developed in Part I of the present paper, which are more important for practical calculations. On the basis of trigonometrical cosines and sinuses the system of orthogonal on the ellipse contour functions $ce_i(t)$ and $se_i(t)$ are introduced. The application of these functions together with the technique of weight function method allows the general procedure for determination of terms of the weight function expansion and then for obtaining the fundamental solution of the elasticity theory to be proposed. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Very often the stress intensity factor is the ultimate goal for practical elastic calculation of the cracked body. But the literature of stress intensity factors is based to a considerable extent on general methods of elasticity, which are complicated and provide more information than we want (see, Bueckner, 1987). The alternative special methods (see, for example, Bueckner, 1977) make use of a particular property of a weight function, which were introduced in the works of Bueckner (1970) and Rice (1972). The weight function allows to calculate stress intensity factors by simple integration over the crack area.

Write a fundamental solution for the weight function Φ for a penny-shaped crack in an infinite body (see, Galin, 1953):

$$\Phi = \frac{\sqrt{R^2-r^2}}{\pi\sqrt{\pi R}(R^2-2Rr\cos(\theta-\varphi)+r^2)} \tag{1}$$

where R is the radius of the circle; θ is the angular coordinate of the crack contour point considered; φ, r are the polar coordinates of the point of the concentrated force application. By the expansion of (1) into Fouries series in terms of θ, it can be obtained (see, for example, Panasyuk, 1968):

$$\Phi = \frac{1}{\pi\sqrt{\pi R}\sqrt{R^2-r^2}}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\left(\frac{r}{R}\right)^n\cdot\cos n\theta\cdot\cos n\varphi+\left(\frac{r}{R}\right)^n\cdot\sin n\theta\cdot\sin n\varphi\right)\right] \tag{2}$$

The availability of formula (1) or (2) for the weight function Φ allows ready determination of the K_I value with any law of the load application:

$$P = P(x,y) = P(r,\varphi) \tag{3}$$

where x, y are Cartesian coordinates in the crack plane. The K_I value is found by simple integration with respect to the crack area S:

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$$K_{\rm I} = \iint\limits_{(s)} \Phi \cdot P(r,\varphi)\,{\rm d}s \tag{4}$$

It is obvious that the availability of the weight function, firstly, simplifies appreciably the procedure of the $K_{\rm I}$ determination for the given law of loading $P(x_1, y_1)$, and secondly, does not restrict the load application law to the polynomial type alone. The third important result of the weight function availability is in the possibility of constructing a fundamental solution of the elasticity theory. For this purpose we shall write Rice's integral formula (see, Rice, 1972), taking as the first state the loading of the crack opposite faces by a pair of unit concentrated forces in the same point (x, y) (or in the polar coordinates (r, φ)) (see, Orynyak, 1998):

$$\int_{\Gamma} F(r,\varphi,t)\mu_s \bar{K}_{\rm I}^2\,{\rm d}\Gamma(t) = \frac{\delta U^2}{\delta T_s}\frac{1}{\pi \cdot a}\cdot H \tag{5}$$

where Γ is the crack contour; H is the Young modulus; $\bar{K}_{\rm I}^2$ is the dimensionless value of the stress intensity factor, $K_{\rm I}^2$, which corresponds to the mutual displacements of the opposite points on the crack faces, U^2:

$$\bar{K}_{\rm I}^2 = \frac{K_{\rm I}^2}{\sqrt{\pi \cdot a}\,\Pi^{1/4}(t)} \tag{6}$$

where t is the parametric angle of the ellipse contour point, and

$$\Pi(t) = \sin^2(t) + \lambda^2\cos^2(t) \tag{7}$$

where $\lambda = a/b \leqslant 1$; a and b are the smallest and the biggest axes of the ellipse, respectively. Function $F(r, \varphi, t)$ is the dimensionless value of the weight function

$$F = \frac{\Phi(r,\varphi,t)}{\sqrt{\pi \cdot a}\,\Pi^{1/4}(t)} \tag{8}$$

and finally, δT_s and μ_s are the characteristic parameter and the coefficient of the s-translations, respectively; s is the translation number (see, Orynyak, 1998). Then, taking $s = 0$ as the translation number, and loading by a pair of unit forces applied in the point (x_1, y_1) or in the polar coordinates (r_1, φ_1) as the second state, from (5) we get:

$$\int_{\Gamma} F(r,\varphi,t)F(r_1,\varphi_1,t)\,{\rm d}\Gamma = \frac{\delta U_\Phi(r,\varphi,r_1,\varphi_1)\cdot H}{\delta a\pi \cdot a} \tag{9a}$$

where U_Φ is the fundamental solution of the elasticity theory. For a circumference, considering (2) and (8), eqn (9a) will take the form:

$$\frac{\delta U_\Phi}{H \cdot \delta a} = \frac{1}{\sqrt{a^2 - r^2}\cdot\sqrt{a^2 - r_1^2}\cdot \pi^2}\left\{2 + \sum_{n=1}^{\infty}\left(\frac{rr_1}{a^2}\right)^n \cos n(\varphi_1 - \varphi)\right\} \tag{9b}$$

Integrating expression (9b) with respect to a, we get:

$$U_{\Phi}(r,\varphi,r_1,\varphi_1) = \frac{1}{\pi^2 Ha}\cdot\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{rr_1}{a^2}\right)^n g_n(rr_1)\cos n(\varphi_1-\varphi) \tag{9c}$$

where

$$g_n = \int_{\max(r/a,r_1/a)}^{1}\frac{\mathrm{d}t}{t^{2n}\sqrt{t^2-(r_1/a)^2}\sqrt{t^2-(r/a)^2}}$$

and ε_n is the Neumann number ($\varepsilon_0 = 1$, $\varepsilon_n = 2$ at $n \geqslant 1$).

The result presented by eqn (9c) corresponds completely to the solution obtained by Martin (1982). Thus, the availability of the weight function makes it possible to obtain also the fundamental solution of the elasticity theory with the use of Rice's integral formula.

The objective of the present work is the construction of the weight function itself rather than of specific solutions for the displacement field as in Part I. Since the mathematical procedure for the integration of the functions of type (1) (with respect to the contour and the area of the ellipse) has not been developed, we have to seek the weight function for an elliptic crack in the form of the expansion into a power series i.e. in the form analogous to (2).

2. GENERAL FORM OF THE FUNCTION $F(r,\varphi,t)$ SOUGHT

For the force application point (x,y) in the polar coordinates (r,φ), introduce parametric coordinates (ρ,ψ) where ρ, ψ are the radius and the angle, respectively. Recall that:

$$\rho = \frac{r}{R(\varphi)},\quad \operatorname{tg}\varphi = \lambda\operatorname{tg}\psi \tag{10}$$

where

$$R(\varphi) = \frac{a}{\sqrt{\sin^2\varphi+\lambda^2\cos^2\varphi}} \tag{11}$$

It is convenient to represent $F(\rho,\psi,t)$ sought in the form of the following trigonometric series:

$$F\sqrt{1-\rho^2}\cdot a^2 = \sum_{i=1}^{\infty}A_i(\rho,\varphi)ce_i(t)+\sum_{i=1}^{\infty}B_i(\rho,\varphi)se_i(t) \tag{12}$$

where factor $\sqrt{1-\rho^2}\cdot a^2$ has been introduced for convenience and A_i and B_i are some unknown functions. Here we introduce the orthogonal functions $ce_i(t)$ and $se_i(t)$ with weight $\Pi^{1/2}(t)$ which are constructed on the basis of trigonometric cosines and sines, respectively, i.e.:

$$(ce_i(t),ce_j(t)) = \int_0^{2\pi}ce_i(t)ce_j(t)\sqrt{\sin^2 t+\lambda^2\cos^2 t}\,\mathrm{d}t = \begin{cases}0 & \text{where } i\neq j\\ 1 & \text{where } i=j\end{cases}$$

$$(se_i(t),se_j(t)) = \int_0^{2\pi}se_i(t)se_j(t)\sqrt{\sin^2 t+\lambda^2\cos^2 t}\,\mathrm{d}t = \begin{cases}0 & \text{where } i\neq j\\ 1 & \text{where } i=j\end{cases}$$

$$(se_i(t),ce_j(t)) = \int_0^{2\pi}se_i(t)ce_j(t)\sqrt{\sin^2 t+\lambda^2\cos^2 t}\,\mathrm{d}t = 0 \tag{13}$$

For the given system of linearly independent elements $\sin nt$ and $\cos nt$, the orthogonal system $ce_n(t)$ and $se_n(t)$ is constructed by the known procedure of Schwarts (see, for example, Lazorkin, 1981). Considering the symmetry of the weight $\Pi^{1/2}(t)$, we obtain that the system of elements $ce_n(t)$ and $se_n(t)$ is divided into four subsystems:

$$1.\ ce_0(t) = \frac{\cos 0\cdot t}{\|\cos 0\cdot t\|} = \frac{1}{\sqrt{4E(k)}}$$

$$ce_2(t) = \frac{\cos 2t - ce_0(t)\int_t \cos 2t ce_0(t)\Pi^{1/2}(t)\,dt}{\|\cos 2t - ce_0(t)\int_t \cos 2t ce_0(t)\Pi^{1/2}(t)\,dt\|}$$

$$ce_{2n}(t) = \frac{\cos 2nt - \sum_{k=0}^{n-1} ce_{2k}(t)\int_t \cos 2nt\cdot ce_{2k}(t)\Pi^{1/2}(t)\,dt}{\left\|\cos 2nt - \sum_{k=0}^{n-1} ce_{2k}(t)\int_t \cos 2nt\cdot ce_{2k}(t)\Pi^{1/2}(t)\,dt\right\|} \tag{14a}$$

$$2.\ ce_1(t) = \frac{\cos t}{\|\cos t\|} = \frac{\cos t(3k^2)^{1/2}}{\{4[(2k^2-1)E(k)-(1-k^2)K(k)]\}^{1/2}}$$

$$ce_{2n+1}(t) = \frac{\cos(2n+1)t - \sum_{k=0}^{n-1} ce_{2k+1}(t)\int_t \cos(2n+1)t ce_{2k+1}(t)\Pi^{1/2}\,dt}{\left\|\cos(2n+1)t - \sum_{k=0}^{n-1} ce_{2n+1}(t)\int_t \cos(2n+1)t ce_{2k+1}(t)\Pi^{1/2}\,dt\right\|} \tag{14b}$$

$$3.\ se_1(t) = \frac{\sin t}{\|\sin t\|} = \frac{\sin t(3k^2)^{1/2}}{\{4[(k^2+1)E(k)-(1-k^2)K(k)]\}^{1/2}}$$

$$se_{2n+1}(t) = \frac{\sin(2n+1)t - \sum_{k=0}^{n-1} se_{2k+1}(t)\int_t \sin(2n+1)t se_{2k+1}(t)\Pi^{1/2}\,dt}{\left\|\sin(2n+1)t - \sum_{k=0}^{n-1} se_{2k+1}(t)\int_t \sin(2n+1)t se_{2k+1}(t)\Pi^{1/2}\,dt\right\|} \tag{14c}$$

$$4.\ se_2(t) = \frac{\sin 2t}{\|\sin 2t\|} = \frac{\sin 2t(15k^4)^{1/2}}{4[2(k^4-k^2+1)E(k)-(1-k^2)(2-k^2)K(k)]^{1/2}}$$

$$se_{2n+2}(t) = \frac{\sin(2n+2)t - \sum_{k=0}^{n-1} se_{2k+2}(t)\int_t \sin(2n+2)t se_{2k+2}(t)\Pi^{1/2}\,dt}{\left\|\sin(2n+2) - \sum_{k=0}^{n-1} se_{2k+2}(t)\int_t \sin(2n+2)t se_{2k+2}(t)\Pi^{1.2}\,dt\right\|} \tag{14d}$$

where $E(k)$ and $K(k)$ are complete elliptic integrals of the II and I kind, respectively, and $k^2 = 1-\lambda^2$, and symbol $\|\mu(t)\|$ corresponds to the designations of the norm of element $\mu(t)$ adopted in mathematics, i.e.

$$\|\mu(t)\| = \left(\int_0^{2\pi} \mu(t)\mu(t)\Pi^{1/2}\,dt\right)^{1/2} = ((\mu(t), \mu(t)))^{1/2}$$

The constructed system of functions (14) is orthonormalized, i.e. the norm of each of the elements $ce_i(t)$ and $se_i(t)$ is equal to unity.

Introducing the following designations:

$$\beta_{ij} = \int_0^{2\pi} \cos itce_j(t)\Pi^{1/2}\,dt$$

$$v_{ij} = \int_0^{2\pi} \sin itse_j(t)\Pi^{1/2}\,dt \tag{15}$$

the formulas (14) can be presented in the more convenient form

$$ce_i(t) = \frac{\cos it - \sum_{j=i-[i/2]2}^{j=i-2} \beta_{ij} ce_j(t)}{\beta_{ii}}$$

$$se_i(t) = \frac{\sin it - \sum_{j=i-[i/2]2}^{j=i-2} v_{ij} se_j(t)}{v_{ii}} \tag{16}$$

where $[i/2]$ is equal to the whole part of the $i/2$ magnitude

Note that the representation of the F expansion in the form of four branches is directly related to formula (38) of Part I of the present paper (see, Orynyak, 1998).

3. THE ESSENCE OF THE METHOD

Consider some law of crack faces displacement which according to Dyson's theorem (see, Dyson, 1891) corresponds to polynomial loading. Let:

$$U(\rho,\psi) = \frac{2\Omega(\rho)a}{H}\left(\frac{x}{b}\right)^k \tag{17a}$$

where k is the integer and $\Omega(\rho) = \sqrt{1-\rho^2}$. Then dimensionless stress intensity factor, $\bar{K}_{\mathrm{I}}$, is equal to:

$$\bar{K}_{\mathrm{I}} = \cos^k \psi \tag{17b}$$

Applying zero translation to displacement (17a) accordingly to formula (11a) of Part I of the present paper (see, Orynyak, 1998) and inserting all necessary parameters into Rice's formula (5) (μ_0 is equal to one), we can obtain:

$$\frac{\pi}{2\lambda\Omega}\int_t \sum_{i=0}^{i=k} A_i(\rho,\psi)ce_i(t)\cos^k t\Pi^{1/2}(t)\,dt = \frac{\rho^k\cos^k\psi}{\Omega(\rho)} + \Omega(\rho)\sum_{m+n=k-2[k/2]}^{m+n=k-2} \tilde{a}_{m,n}\left(\frac{x}{b}\right)^m\left(\frac{y}{a}\right)^n \tag{18}$$

where $\tilde{a}_{m,n}$ are some coefficients.

As it follows from (18) with accounting for (13) and (16) the function A_n sought can be written in general form:

$$\frac{\pi}{2\lambda}A_n(\rho,t) = ce_n(t)\rho^n + (1-\rho^2)\{ce_{n-2}(t)\rho^{n-2}a^n_{n-2,0}$$
$$+ce_{n-4}(t)\rho^{n-4}(a^n_{n-4,2}\rho^2 + a^n_{n-4,0}) + ce_{n-6}(t)\rho^{n-6}(a^n_{n-6,4}\rho^4 + a^n_{n-6,2}\rho^2 + a^n_{n-6,0}) + \cdots\} \tag{19}$$

Similarly to the foregoing, we can write:

$$\frac{\pi}{2\lambda}B_n(\rho,t) = se_n(t)\rho^n+(1-\rho^2)\{se_{n-2}(t)\rho^{n-2}b^n_{n-2,0}$$

$$+se_{n-4}(t)\rho^{n-4}(b^n_{n-4,2}\rho^2+b^n_{n-4,0})+se_{n-6}(t)\rho^{n-6}(b^n_{n-6,4}\rho^4+b^n_{n-6,2}\rho^2+b^n_{n-6,0})+\cdots\} \quad (20)$$

There are $[(k-1)k/2]$ unknown coefficients $a^n_{i,j}$ (also $b^n_{i,j}$), where $k=[n/2]$, presented in the general forms (19) and (20). To determine them we write the following condition and name it as condition C.

If the polynomial loading $\rho_{ij}(x,y)$ is applied to crack faces

$$\rho_{ij}(x,y) = \left(\frac{x}{b}\right)^j\left(\frac{y}{a}\right)^i = \rho^{i+j}\cos^j\psi\sin^i\psi \quad (21)$$

then resulting contribution to K_I accordingly to formula (4) from terms A_m and B_m, where $m>i+j$, should be equal to zero (otherwise, at loading (21) the resulting K_I will contain the terms $\cos mt$ and $\sin mt$ which contradicts the Dyson's theorem).

When calculating the integrals of type (4) it is convenient to go over to parametric coordinates. Then the element of area $\mathrm{d}s$ equals to:

$$\mathrm{d}s = \frac{a^2}{\lambda}\rho\,\mathrm{d}\rho\,\mathrm{d}t \quad (22)$$

Substituting (22) into (4) and considering (8) and (12) we get:

$$K_\mathrm{I}(t) = \frac{\Pi^{1/4}\sqrt{\pi a}}{\lambda}\int_0^{2\pi}\int_0^1\frac{P(\rho,\psi)}{\sqrt{1-\rho^2}}\left(\sum_{i=0}^{\infty}A_i(\rho,\psi)ce_i(t)+\sum_{i=0}^{\infty}B_i(\rho,\psi)se_i(t)\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\psi \quad (23)$$

The unknown coefficients $a^n_{i,j}$ are determined from the conditions C, i.e. $(k-1)k/2$ of the unknown coefficients $a^n_{i,j}$ (where $k=[n/2]$, i.e. k is equal to the whole part of the $n/2$ magnitude) are defined from $(k-1)(k/2)$ equations of the form:

$$\int_0^{2\pi}\int_0^1 A_n(\rho,t)\cdot P_{ij}(\rho,\psi)\frac{\rho}{\sqrt{1-\rho^2}}\mathrm{d}\rho\,\mathrm{d}t = 0 \quad (24)$$

where for convenience the elements $P_{ij}(\rho,\psi)$ of matrix of loading are taken in the form of linear combination of (21):

$$\begin{array}{l}\rho^{n-2}\cos(n-2)t,\quad \rho^{n-2}\cos(n-4)t,\ldots,\rho^{n-2}\cos\left(n-2\left[\frac{n}{2}\right]\right)t\\ \rho^{n-4}\cos(n-4)t,\quad \rho^{n-4}\cos(n-6)t,\ldots,\rho^{n-4}\cos\left(n-2\left[\frac{n}{2}\right]\right)t\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \rho^{(n-2[n/2])}\end{array} \quad (25)$$

Accounting for (19) and substituting into eqn (24) first $P=\rho^{n-2}\cos(n-2)t$ define the value of $a^n_{n-2,0}$; then substituting in turn two loads proportional to $\cos(n-4)t$, i.e. $\rho^{n-2}\cos(n-4)t$ and $\rho^{n-4}\cos(n-4)t$ find the coefficients occurring in eqn (19) at $ce_{n-4}(t)$, i.e. $a^n_{n-4,2}$ and $a^n_{n-4,0}$. Then to determine three coefficients found with $ce_{n-6}(t)$ in expression (19), substitute three loads proportional to $\cos(2n-6)t$ into eqn (24), and so on. Analogous procedure can be suggested for determination of the coefficients $b^n_{i,j}$. In this case A_n should be substituted by B_n in formula (24) and functions cos should be substituted by functions sin in (25).

Note that to determine A_n and B_n there is no need to know the A_i and B_i values of a lower order.

4. DETERMINATION OF THE INITIAL FUNCTIONS A_i AND B_i OF THE F EXPANSION

As it immediately follows from the general forms (19) and (20) the first functions of each branch of the F expansion contain only one term:

$$A_0 = \frac{2\lambda}{\pi} ce_0(t); \quad A_1 = \frac{2\lambda\rho}{\pi} ce_1(t) \tag{26a–b}$$

$$B_1 = \frac{2\lambda\rho}{\pi} se_1(t); \quad B_2 = \frac{2\lambda\rho^2}{\pi} se_2(t) \tag{26c–d}$$

To verify the correctness of the obtained functions (26) integrate the weight function in accordance with (23) for the following loading:

$$P_{0,0} = 1; \quad P_{0,1} = \rho\cos\psi = \frac{x}{b}$$

$$P_{1,0} = \rho\sin\psi = \frac{y}{a}; \quad P_{1,1} = \rho^2 \sin\psi\cos\psi = \frac{y}{a}\frac{x}{b} \tag{27}$$

Thus, substituting (26) and (27) into (23) the following K_{I} values we get:

$$K_{\mathrm{I}}^{0,0} = \frac{\sqrt{\pi a}\Pi^{1/4}\cdot 4ce_0(t)}{\|\cos 0t\|} = \frac{\sqrt{\pi a}\Pi^{1/4}(t)}{E(k)}$$

$$K_{\mathrm{I}}^{0,1} = \frac{\sqrt{\pi a}\Pi^{1/4}(t)4ce_1(t)}{3\|\cos t\|} = \frac{\sqrt{\pi a}\Pi^{1/4}\cdot\cos t\cdot k^2}{(2k^2-1)E(k)-(1-k^2)K(k)}$$

$$K_{\mathrm{I}}^{1,0} = \frac{\sqrt{\pi a}\Pi^{1/4}(t)4se_1(t)}{2\|\sin t\|} = \frac{\sqrt{\pi a}\Pi^{1/4}\cdot\sin t\cdot k^2}{(k^2+1)E(k)-(1-k^2)K(k)}$$

$$K_{\mathrm{I}}^{1,1} = \frac{\sqrt{\pi a}\Pi^{1/4}(t)16se_2(t)}{15\|\sin 2t\|} = \frac{\sqrt{\pi a}\Pi^{1/4}\cdot\sin 2t\cdot k^4}{[2(k^4-k^2+1)E-(1-k^2)(2-k^2)K(k)]} \tag{28}$$

The K_{I} values determined agree completely with the known solutions of Shah and Kobayashi (1971).

As another example let us consider function $A_2(\rho,\psi)$ of the first branch. In accordance with formula (19), we shall write in a general form:

$$\frac{\pi}{2\lambda}A_2(\rho,t) = ce_2(t)\rho^2 + (1-\rho^2)a_{0,0}^2 ce_0(t) \tag{29a}$$

In accordance with (24), the following integral is equal to zero:

$$\int_0^{2\pi}\int_0^1 \left\{\left(\frac{\cos 2t - ce_0(t)\beta_{2,0}}{\beta_{2,2}}\right)\rho^2 + (1-\rho^2)a_{0,0}^2 ce_0(t)\right\}\frac{\rho\,\mathrm{d}\rho\,\mathrm{d}t}{\sqrt{1-\rho^2}} = 0$$

from which it follows, considering the permutability of the order of integration:

$$a_{0,0}^{2} = \frac{2\beta_{2,0}}{\beta_{2,2}} \tag{30a}$$

Similarly, we can get the expression for the second function of the expansion of each branch. In succession we have:

$$\frac{\pi}{2\lambda} A_3(\rho, t) = ce_3(t)\rho^3 + \rho(1-\rho^2)a_{1,0}^{3} ce_1(t) \tag{29b}$$

In accordance with (24), the contribution of expression (29b) into the K_{I} value for the load $P_{0,1}(x, y) = \rho \cos \psi$ is equal to zero: therefore

$$a_{1,0}^{3} = \frac{4\beta_{3,1}}{\beta_{3,3}} \tag{30b}$$

Now we write:

$$\frac{\pi}{2\lambda} B_3(\rho, t) = se_3(t)\rho^3 + \rho(1-\rho^2)b_{1,0}^{3} se_1(t) \tag{29c}$$

where coefficient $b_{1,0}^{3}$ is determined in accordance with formula (24) at the load $P_{1,0}(x, y) = \rho \sin \psi$ and is equal to:

$$b_{1,0}^{3} = \frac{4\nu_{3,1}}{\nu_{3,3}} \tag{30c}$$

and finally:

$$\frac{\pi}{2\lambda} B_4(\rho, t) = se_4(t)\rho^4 + \rho^2(1-\rho^2)b_{2,0}^{4} se_2(t) \tag{29d}$$

Function B_4 makes a zero contribution into K_{I} at the load $P_{1,1}(x, y) = \rho^2 \sin \psi \cos \psi$, from which we get:

$$b_{2,0}^{4} = \frac{6\nu_{4,2}}{\nu_{4,4}} \tag{30d}$$

The simplicity of the method is surprising even for us. We can only note that obtaining the second terms of the four branches of the expansion A_2, A_3, B_3, B_4 took us not more than 10 minutes! And the first and the second terms allow ready determination of K_{I} for the loads of the third degree and lower, i.e. they make it possible to solve easily the problem accessible so far only to few scientists. And the availability of B_2 and B_4 values allows K_{I} determination at the loads of the fourth degree $P_{31}(x, y) = xy^3$ and $P_{13}(x, y) = yx^3$.

To complete the problem of K_{I} determination at the loads of the fourth degree and to clarify the procedure of getting A_i and B_i of a higher order, we shall get the A_4 value. Write in a general form:

$$\frac{\pi}{2\lambda} A_4(\rho, t) = ce_4(t)\rho^4 + ce_2(t)\rho^2(1-\rho^2)a_{2,0}^{4} + ce_0(t)(1-\rho^2)(a_{0,2}^{4}\rho^2 + a_{0,0}^{4}) \tag{31}$$

Substituting the value $P_3 = \rho^2 \cos 2\psi$, and function (31) into (24) considering that nonzero values during integration with respect to t yields only function $ce_2(t)$, we get:

$$a_{2,0} = \frac{4\beta_{4,2}}{\beta_{4,4}} \tag{34a}$$

Then, substituting the values $P_2 = \rho^2$ and $P_1 = 1$ into expression (24), and considering the expansion of the function $ce_2(t)$ in terms of functions $\cos 2t$ and $ce_0(t)$, we obtain:

$$a^4_{0,2} = -\frac{23}{3}\frac{\beta_{4,2}\beta_{2,0}}{\beta_{4,4}\beta_{2,2}};\quad a^4_{0,0} = -\frac{8}{3}\frac{\beta_{4,2}\beta_{2,0}}{\beta_{4,4}\beta_{2,2}} \tag{34b–c}$$

5. FUNDAMENTAL SOLUTION OF THE ELASTICITY THEORY

In the work of Altluri and Nishioka (1991) the solutions for displacement field $U(x, y)$ were initially obtained and after the partial differentials $\mathrm{d}U/\mathrm{d}a$ were calculated in order to obtain the integrally averaged weight function. Here the inverse procedure of obtaining the partial differentials $\mathrm{d}U/\mathrm{d}a$ from initially calculated weight function and subsequent determination of the displacement field is demonstrated.

Substitute the expansion for function F into (9a). Considering orthonormalization of functions ce_i and se_i, we get:

$$\frac{\delta U_\phi}{\delta a} = w_0 = \frac{\pi a}{H}\cdot\frac{a}{\lambda}\cdot\frac{1}{a^4\sqrt{1-\rho^2}\sqrt{1-\rho_1^2}}\left[\sum_{i=0}^{\infty} A_i(\rho,\psi)A_i(\rho_1,\psi_1)+\sum_{i=0}^{\infty} B_i(\rho,\psi)B_i(\rho_1,\psi_1)\right] \tag{35a}$$

Here $(\delta U_\phi/\delta a) = w_0$ is the variation of the displacement field during zero translation of the crack front, i.e. when $\delta a = \lambda\delta b$, $\lambda = \text{const}$. Therefore, taking into account (10) and (11) rewrite eqn (35a) in the form:

$$\frac{\delta U_\phi}{\delta a} = w_0 = \frac{\pi}{H\lambda} \times \frac{\left[\sum_{i=0}^{\infty} A_i\left(\frac{rf(\psi)}{a},\psi\right)A_i\left(\frac{r_1 f(\psi_1)}{a},\psi_1\right)+\sum_{i=0}^{\infty} B_i\left(\frac{rf(\psi)}{a},\psi\right)B_i\left(\frac{r_1 f(\psi_1)}{a},\psi_1\right)\right]}{\sqrt{a^2-r^2f^2(\psi)}\sqrt{a^2-r_1^2f^2(\psi_1)}} \tag{35b}$$

Integrating eqns (35b) with respect to a from $a = \max\{r\cdot f(\psi), r_1\cdot f(\psi_1)\}$ to a like (9c), we obtain the fundamental solution of the elasticity theory. We shall not present the expressions for U_Φ. Note only that the integral eqn (35b) can be taken and be expressed by elliptic integrals of the I and the II kind.

6. CONCLUSIONS

(1) A general procedure has been proposed for finding the expansion of a weight function into a series and specific values of the terms of such expansion have been obtained, which yield exact values of the stress intensity factors for the loading up to the fourth degree inclusive.
(2) A general expression has been obtained in quadratures for the fundamental solution of the elasticity theory in the case of the action of a concentrated force in an arbitrary point of the crack surface.

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